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# Testing Desings for Identifying at Most Two Defectives and Their Optimalities of $\lambda -$Singly Linked Block Designs through Non-Adaptive Hypergeometric Group

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***Abstract*: In this paper we develop a new method of construction of $\lambda -$singly linked block designs through non-adaptive Hypergeometric group testing designs for identifying at most two defectives for $V^* \equiv 0 (\mathrm{mod}\ 6)$ and $V^* \equiv 2 (\mathrm{mod}\ 6)$. We have shown that such designs are Type I designs and Geometry designs.**

***Key words and phrases*: Non-adaptive Hypergeometric group testing designs, $\lambda -$linked block designs, Geometry designs, multigraph designs, Type I optimality.**

## I. INTRODUCTION

The technique of group testing was originally proposed by Dorfman (1943) in the context of blood testing. The common assumption in a group testing problem is that there is no test error. The group testing problem has been studied by several authors viz: Sobel and Groll (1950, 1966), Hwang and Sos (1981), Weidenman (1984) and Weidenman and Raghavarao (1987a,b). Dorfman's (1943) original procedure has been modified and extended to other screening situations. The group testing problem is the classification of each of n items into one of two disjoint categories called defective and non-defective in a series of $V^*$ or t test, where a subset of items is used in each test. Each group test can provide only one of two outcomes, a positive outcome indicates the presence of one or more defective items among the items tested while a negative outcome indicates no defective items among the tested items.

The non-adaptive hypergeometric problem was first introduced by Hwang and Sos (1981). This problem arises when the number of defectives is known prior to testing. It can be approached in two ways. The traditional approach in the literature has focused on minimizing the number of tests required to identify the defectives for a given number of items. The second approach introduced by Weideman and Raghavarao (1987a) is to maximize the number of items that can be tested in a given number of tests. Weideman and Raghavarao (1987b) considered the problem of constructing group testing designs with n items that can be identify at most two defectives by performing $V^*$ tests through the dual formulation of the design. Ghosh and Thannippara (1991) developed some more such designs from Hyper-cubic designs and BIBD.

## II. λ-SINGLY LINKED BLOCK DESIGNS

Bose (1963, 1975, 1976) studied the problem of designs and multigraphs. His investigation was based on BIBD, Lattice designs, Mutually orthogonal latin square designs etc. In this investigation, Bose had introduced the definitions of $\lambda -$linked block designs, block multi-graph designs, treatment multi-graph designs and geometry designs and their constructions. From the lines of Bose (1975) we state the following definitions.

***$\lambda$ −Linked block designs:***

Given a design d, we can obtain another design $d^*$, the dual of d by interchanging the blocks and treatments. Thus the treatments $t_1, t_2, \dots, t_v$ of d becomes blocks of $d^*$, and the blocks $B_1, B_2, \dots, B_b$ of d become treatments of $d^*$. Actually this process is known as dualization. If the treatment $\theta_i$ of d occurs in the block $B_i$ of d then the treatment $B_i$ of $d^*$ occurs in the block $\theta_i$ of $d^*$. Thus the dual of a BIBD with parameters $(\upsilon, b, r, k, \lambda)$ is configuration $(b, \upsilon, k, r)$ in which any two blocks intersect in $\lambda$ treatments. This is called $\lambda$ −linked block design. In particular, for $\lambda = 1$, the design is called a singly linked block (SLB) design.

***Geometry designs:***

A design will be called a geometry if any two treatments of d cannot occur together in more than one block. In this case the treatments may be called the points, and the blocks may be called the lines of the geometry.

***Treatment multigraph and Block multigraph designs:***

Given a design d, we can obtain from it two multigraphs, the treatment multigraph $G(d)$ and the block multigraph $G^*(d)$. The treatment of the design can be obtained in the following manner. Let, the two treatments $t_i$ and $t_\alpha$ of d occur together in exactly $\lambda_{i\alpha}$ blocks then $m(t_i, t_\alpha) = \lambda_{i\alpha}$ where m represents the multiplicity function. In particular if d is geometry, then $\lambda_{i\alpha} = 0$ or 1. Hence $G(d)$ is graph. A graph G is a multigraph in which the multiplicity function can take only two values 0 and 1.

Similarly, we can define the block multigraph also. If the blocks $B_j, B_\beta$ of d intersect in exactly $\mu_{j\beta}$ treatments then $m(B_j, B_\beta) = \mu_{j\beta}$.

## III. TYPE I OPTIMALITY OF GROUP TESTING DESINGS FOR $V^* \equiv 0 (\bmod 6)$ AND $V^* \equiv 2 \ (\bmod 6)$

Blocking is an experimental technique commonly used in agricultural, industrial and biological experiments to eliminate heterogeneity in one direction. In any experimental situation requiring usage of a block, it is desirable to maximize the amount of information gained on the treatments being studied by using an 'Optimal Block Design'. Several results are known concerning the type I optimality of block designs in class $d(\upsilon, b, k)$. For example, it is well known that a BIBD is optimal in $d(\upsilon, b, k)$ under all type I optimality criteria. Certain types of regular graph designs (RGD's) which are not BIBD's have also been shown to be optimal under various type I criteria in a number of classes and sub classes of $d(\upsilon, b, k)$, eg. See Cinniffe and Stone (1975); Shah, Raghavarao and Khattri (1976) ; Williams, Patterson and John (1977) ; Cheng (1978,1979). However those designs which are type I optimal in a vast majority of classes $d(\upsilon, b, k)$ remain unknown.

Jacroux (1985) studied the type I optimality of semi-regular design (SRGD), we call d a SRGD if d is binary and if $N_d N'_d$ has all of its diagonal elements and off diagonal elements differing by at most one (see Jacroux, 1985). The notion of an SRGD is a generalization of the definition given by Mitchell and John (1977) for a regular graph design (RGD) and reduces to their definitions when $bk/\upsilon$ is an integer. If d is an RGD and its associations matrix has the additional property that all of its off-diagonal elements are equal, then d is called Balanced Incomplete Block Design (BIBD).

In this paper we consider the determination of type I optimal block designs from $V^* \equiv 0(\bmod 6)$ and $V^* \equiv 2 \ (\bmod 6)$ group testing designs having $\upsilon$ treatment arranged in b blocks of size k. For simplicity, it is assumed throughout in this section that $\upsilon > k$.

Let d be a block design such as described above. Then d has associated with its $\upsilon \times b$ incidence matrix $N_d$ whose entries $n_{ij}$ give the number of times the ith treatment occurs in jth block. In case of $n_{ij} = 1$ or 0 for all i and j, the design is called a binary design. The ith row sum of $N_d$ is denoted by $r_i$ and represents the number of times the treatment i is replicated in the design. The matrix $N_d N'_d$ where $N'_d$ is the transpose of $N_d$, is referred to as the concurrence matrix of d, and its entries are denoted by $\lambda_{ij}$. Hence we consider the design under the two-way additive model. A design d is connected if and only if its C −matrix has rank $(\nu - 1)$.

For the class $d(\upsilon, b, k)$, let $Z_{d0} = 0 < Z_{d1} \le Z_{d2} \le \cdots \le Z_{d(\nu-1)}$ denote the eigen values of the associated C −matrix. A design is said to be $\phi_f$-optimal in $d(\upsilon, b, k)$ provided

$$\phi_f(C_d) = \sum_{i=1}^{v-1} f(Z_{di}) \quad (1)$$ is minimal over all designs $d(\upsilon, b, k)$ where f is non-increasing and convex real valued function. The well-known A and D criteria correspond to taking $f(x) = \frac{1}{x}$ and $-\log x$ in equation (1) respectively.

***Type* 1 *optimality***: $\phi_f$ of the equation (1) is called type 1 optimality criterion if f satisfies the following conditions:

(i) f is continuously differentiable on $(0, \max \operatorname{tr}\{C_d\})$ and

$f' < 0, f'' < 0, f''' < 0$ on $(0, \max \operatorname{tr}\{C_d\})$.

(ii) f is continuous at 0 or $\lim_{x\to 0} f(x) = \infty$.

Note that the A and D mentioned above are type 1 optimality criteria. This definition is due to Jacroux (1985).

***Preliminary results*:**

***Lemma* 1.** A group testing design d for identifying at most two defectives exists. Then it satisfies the following conditions.

(i) $B_i^* \cup B_j^* = B_k^* \cup B_l^*; i, j, k, l = 1, 2, \dots, n, (i, j) \neq (k, l)$ where $B_i^*$ denotes the number of the tests in d in which the ith item is tested.

(ii) In $d^*$ any pair of treatment can appear at most once.

(iii) $n \leq \left\lfloor \frac{v(v+1)}{6} \right\rfloor$ where $[\cdot]$ denotes the greatest integer.

The conditions discussed above in Lemma 1 are provided by Weideman and Raghavarao (1987a).

Definition 1. A group testing design, d, is said to be a $\lambda$ −linked block design if the following conditions are satisfied:

(i) $V^* \equiv 0 (\mathrm{mod}\, 6)$ and $V^* \equiv 2\ (\mathrm{mod}\, 6)$ where $V^*$ denotes the number of treatments in the dual design $d^*$.

(ii) All blocks of d have the same size k.

(iii) $r_1^* = r_2^* = \cdots = r_v^* = r^*$ that is, the replication of the test treatments in dual design, $d^*$, remains the same.

(iv) $B_i \cup B_j = B_k \cup B_l$ ; $i, j, k, l = 1, 2, \dots, n, (i, j) \neq (k, l)$ that is any two blocks intersect in $\lambda$ treatments, where $B_i$ denotes the ith block of group testing design, d.

## IV. SOME RESULTS RELATED TO TYPE 1 OPTIMALITY

In this section we are discussing some results related to Type 1 optimality which are proven in Jacroux (1985).

Consider $A = \operatorname{tr} C_d = \sum_{i=1}^{v-1} Z_{di}$, $B = \operatorname{tr} C_d^2 + \min\left(\frac{1}{2}, \frac{4}{k^2}\right)$,

$\operatorname{tr} C_d^2 = \sum_{i=1}^{v-1} Z_{di} \geq B$.

Now , $m_1$ and $m_1^*$ are non-negative constants ( to be defined later) such that

$$(A - m_1)^2 \geq B - m_1 \geq \frac{(A - m_1)^2}{(v-2)},$$

$$(A - m_1^*)^2 \geq B - m_1^* \geq \frac{(A - m_1^*)^2}{(v-2)},$$

$$P_1 = [(B - m_1) - (A - m_1)^2/(v-2)]^{\frac{1}{2}},$$

$$m_2 = [(A - m_1) + \{(v-2)/(v-3)\}^{\frac{1}{2}} P_1]/(v-2)$$

$$m_3 = [(A - m_1) + \{(v-2)(v-3)\}^{\frac{1}{2}} P_1]/(v-2)$$

$$m_4 = [A - (2/k) - m_1^*]/(v-2).$$

***Lemma* 2.** Let $d(\upsilon, b, k)$ such that $\frac{bk}{\upsilon}$ is not an integer and let $d(\upsilon, b, k)$ be an SRGD with $C$ −matrix $C_d$ having non-zero eigen values $Z_{d1} \le Z_{d2} \le \cdots \le Z_{d(\nu-1)}$. Now let

$m_1 = m_1^* = \frac{r(k-1)\nu}{(\nu-1)k}$. If $m_1 \le m_2, m_1^* \le m_4$ and $\sum_{i=1}^{\nu-1} f(Z_{di}) < \min\{f(m_1) + (\nu-3) f(m_2) + f(m_3), f(m_1^*) + (\nu-2) f(m_4)\}$ then a $\phi_f$ −optimal design in $d(\upsilon, b, k)$ must be an SRGD.

***Examples for $\lambda$ −linked Block Designs***

Here we have shown that non-adaptive hypergeometric group designs for identifying two defectives constructed by Weideman and Raghavarao (1987s) from $V^* \equiv 0 (\mathrm{mod} 6)$ and $V^* \equiv 2 (\mathrm{mod} 6)$ are $\lambda$ −linked block designs.

***Example* 1. *(Using definition* 1*)***

Let $V^* = 6$, the blocks of dual design, $d^*$, are $B_1^* = (1,2), B_2^* = (3,4), B_3^* = (5,6)$, $B_4^* = (1,3,5), B_5^* = (1,4,6)$, $B_6^* = (2,3,6), B_4^* = (2,4,5)$.

The blocks of corresponding group testing designs (GTD) are

$$B_1 = (1,4,5), B_2 = (1,6,7), B_3 = (2,4,6),$$

$$B_4 = (2,5,7), B_5 = (3,4,7), B_6 = (3,5,6).$$

In this example we observe that

$$B_i \cap B_j = B_k \cap B_i;\ i, j, k, l = 1,2, \dots, 6, (i, j) \ne (k, l).$$

Clearly, it is also a block multigraph design and geometry design, since $m(B_j, B_\beta) = 1$ for $j, \beta = 1,2, \dots, 6$.

***Example* 2.** ( ***Using Definition* 1**).

Let $V^* = 8$, the blocks of dual design,$d^*$, are

$$B_1^* = (1,2)\ ,\ B_2^* = (3,4)\ ,\ B_3^* = (5,6)\ ,\ B_4^* = (7,8)$$

$$B_5^* = (1,3,5)\ ,\ B_6^* = (2,4,6)\ ,\ B_7^* = (2,3,7)\ ,\ B_8^* = (1,4,8)$$

$$B_9^* = (1,6,7)\ ,\ B_{10}^* = (2,5,8)\ ,\ B_{11}^* = (3,6,8)\ ,\ B_{12}^* = (4,5,7)$$

The blocks of corresponding GTD are

$$B_1^* = (1,5,8,9)\ ,\ B_2^* = (1,6,7,10)\ ,\ B_3^* = (2,5,7,11)$$

$$B_4^* = (2,6,8,12)\ ,\ B_5^* = (3,5,10,12)\ ,\ B_6^* = (3,5,9,11)$$

$$B_7^* = (4,7,9,12)\ ,\ B_8^* = (4,8,10,11)$$

Now note that

$$B_i \cap B_j = B_k \cap B_i;\ i, j, k, l = 1,2, \dots, 6, (i, j) \ne (k, l).$$

Since $\lambda = 1$, GTD is a singly linked block design. Clearly, it is also a block multigraph design and also geometry design, since $m(B_j, B_\beta) = 1$ for $j, \beta = 1,2, \dots, 8$.

***Theorem* 1.** Group testing design, d, for $V^* \equiv 0 (\mathrm{mod} 6)$ and $V^* \equiv 2 (\mathrm{mod} 6)$ implies the existence of a semi-regular graph design.

Proof. If d binary and $N_d N'_d$ has all of its diagonal elements and off diagonal elements differing by at most one, then GTD for $V^* \equiv 0 (\mathrm{mod} 6)$ and $V^* \equiv 2 (\mathrm{mod} 6)$ are SRGD. This completes the proof.

***Theorem 2.*** A BIBD obtained from a GTD (d) for $V^* \equiv 0 (\mathrm{mod} 6)$ by adding a single block.

Proof. If d is an RGD ( i.e, $N_d N'_d$ has all of its diagonal elements equal and off diagonal elements differing by at most one) and its association matrix has the additional property that all of its off diagonal elements are equal, then d is called a balanced incomplete block design (BIBD).

***Lemma 3.*** An SRGD obtained from a BIBD by adding a single block. This lemma is due to Jacroux (1985).

***Theorem 3.*** An SRGD obtained from a GTD for $V^* \equiv 0(\mathrm{mod}6)$ by adding two identical blocks.

Proof. The proof of this Theorem can obtained using Theorem 1 and Lemma 3. Determination of Type I optima design ( Using theorem 1, 2, &3). By using the Theorem 3 we construct the following SRGD from GTD.

Table showing the block of Type I optimal SRGD

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 2 | 3 | 3 | 1 | 1 |
| 4 | 6 | 4 | 5 | 4 | 5 | 2 | 2 |
| 5 | 7 | 6 | 7 | 7 | 6 | 3 | 3 |

From the above table let the design $d_1$ be

| | | | | | |
|---|---|---|---|---|---|
| 1 | 1 | 2 | 2 | 3 | 3 |
| 4 | 6 | 4 | 5 | 4 | 5 |
| 5 | 7 | 6 | 7 | 7 | 6 |

The design $d_2$ be

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 2 | 3 | 3 | 1 |
| 4 | 6 | 4 | 5 | 4 | 5 | 2 |
| 5 | 7 | 6 | 7 | 7 | 6 | 3 |

The design $d_3$ be

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 2 | 3 | 3 | 1 | 1 |
| 4 | 6 | 4 | 5 | 4 | 5 | 2 | 2 |
| 5 | 7 | 6 | 7 | 7 | 6 | 3 | 3 |

$d_1(7,6,3)$ is a GTD and also an SRGD (by Theorem 1). $d_2(7,7,3)$ is a BIBD (by theorem 2). $d_3(7,6,3)$ is an SRGD (by theorem 3).

Consider the design $d_3(7,6,3)$. In this design the value of $\frac{b\nu}{k} = \frac{24}{7}$ which is not an integer, and $\mathrm{tr}(C_d) = 16$, $\mathrm{tr}(C_d^2) = 44$. Note that $d_2(7,7,3)$ is a BIBD. Now let $d_3$ denote the design obtained by adding a single block $(b^*)$ to $d_2$ where $b^*$ is any integer such that $b^*k < \nu$. Let $Z_{d_2}$ denote the common value of the $(\nu - 1)$non-zero eigen values of $C_{d_2,}$ then $C_{d_3}$ has two distinct non-zero eigen values, the larger of which occurs with multiplicity $b^*(k-1)$ and is equal to $Z_{d_2} + 1$ while the smaller is equal to $Z_{d_2}$ and occurs with the multiplicity $\nu - 1 - b^*(k-1)$. Using the result we obtain $Z_{d_{3,1}} = 3.33$ with multiplicity 2 and $Z_{d_{3,2}} = 2.33$ with multiplicity 4. Note the following

$\sum_{i=1}^{6} Z_{d_{3,i}} = 16$, $\prod_{i=1}^{6} Z_{d_{3,i}} = 54.89$, $\sum_{i=1}^{6} \frac{1}{Z_{d_{3,i}}} = 2$, $B = 44.44$ , $m_1 = m_1^* = 2.33$, $P_1 = 1.28$, $m_2 = 2.45$, $m_3 = 3.02$, $m_4 = 2.60$, $\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} = 2.397$, $\frac{1}{m_1} + \frac{5}{m_4} = 2.351$.

It follows from the Lemma 2 that an A- or D-optimal design $d(7,8,3)$ must be an SRGD. It is type-I optimal design, Jacroux (1985).

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